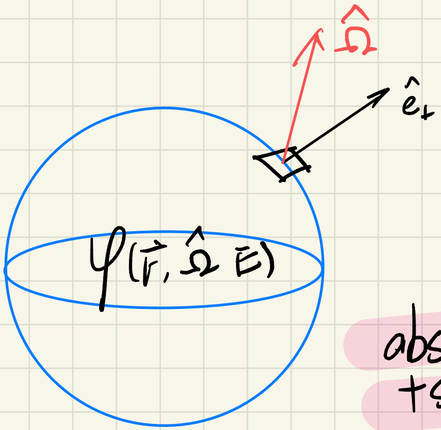
Linear Boltzman Transfer function.

Time independent.
steady state



fluence : $\vec{r}$, $\vec{\Omega}$, E, $\cancel{t}$

net flow: $-\iint_S \hat{\Omega}\, \varphi(\vec{r}, \hat{\Omega}, E) \cdot \hat{e}_\perp$

absorbed + scattered : $-\Sigma_a \varphi(\vec{r}, \hat{\Omega}, E) \underbrace{- \Sigma_c \varphi(\vec{r}, \hat{\Omega}, E)}_{\text{scattered out.}}$

Scattered in : $+\int d\Omega' \int dE'\, \underbrace{\Sigma_s(\hat{\Omega}, \hat{\Omega}', E, E')}_{\text{look up table.}}\, \varphi(\vec{r}, \hat{\Omega}', E')$

Source : $+ S(\vec{r}, \Omega, E)$

$$-\iiint_V \text{absorbed + scattered} + \iiint_V \text{scatter in} + \iiint_V \text{source} - \iint_S \text{net flow} = 0$$

Derivative form. $K_s \varphi(\vec{r}, \hat{\Omega}', E') + S(\vec{r}, \Omega, E) = \Sigma_t \varphi + \hat{\Omega} \vec{\nabla} \varphi$

in = out.

Boundary conditions.

Parametric equation.

Practical Implementation.

	γ	e^-	e^+
γ	Compton/~~Rayleigh~~.	Compton PP. PE	PP.
e^-	~~Bremsstrahlung~~.	Columb. collision	X
e^+	Annihilation.	columb.	X

photon: $$\hat{\Omega}_\gamma \vec{\nabla}\varphi_\gamma + \Sigma_t \varphi^\gamma = K^{\gamma\gamma}\varphi^\gamma \cancel{+K^{e\gamma}\varphi^e} + S^\gamma$$

electron: $$\hat{\Omega}_e \vec{\nabla}\varphi_e + \Sigma_t\varphi^e = \boxed{K^{ee}\varphi^e} + K^{\gamma e}\varphi^\gamma + S^e.$$

soft$(\Delta E<\Delta)$+hard$(\Delta E>\Delta)$ $$\frac{\partial L(E)}{\partial E}\varphi^e + K^{ee}(\Delta E_{th})\varphi^e.$$

* uncollided photon: $$(\hat{\Omega}_\gamma\cdot\vec{\nabla} + \Sigma_t^\gamma)\varphi^\gamma_{unc} = S^\gamma$$

Collided photon: $$(\hat{\Omega}_\gamma\vec{\nabla} + \Sigma_t^\gamma)\varphi^\gamma_{col} = K^{\gamma\gamma}\varphi_{col} + K^{\gamma\gamma}\varphi_{unc}$$

Electron: $$\hat{\Omega}_e\vec{\nabla}\varphi_e + \Sigma_t\varphi_e = \frac{\partial L_\Delta}{\partial E}\varphi^e + K^{ee}_\Delta\varphi^e + K^{\gamma\gamma}\varphi_{unc} + K^{\gamma\gamma}\varphi_{col}$$

* $$\varphi^\gamma_{unc}(r) = S_0\frac{\exp(-\Sigma_t r)}{4\pi r^2}. \rightarrow \varphi_{col} \rightarrow \varphi_e.$$

How to solve LBTE?

$$\hat{\Omega}\cdot\vec{\nabla}\varphi(\vec{r},\hat{\Omega},E)+\Sigma_t(\vec{r},\hat{\Omega},\vec{E})\varphi(\vec{r},\hat{\Omega},E)$$

$$=K_{\vec{r},\hat{\Omega}'\to\hat{\Omega},E'\to E}\varphi(\vec{r},\hat{\Omega},E)+S(\vec{r},\hat{\Omega},E)$$

Step 1. Dimensionality Reduction on $\hat{\Omega}$

① $\varphi(\vec{r},\hat{\Omega},E)=\sum_{l=0}^{N_L}\sum_{m=-l}^{m=l}\varphi_l^m(\vec{r},E)\,Y_l^m(\hat{\Omega})$

unknown. Known.

② $\Sigma_s(\vec{r},\hat{\Omega},\hat{\Omega}',E,E')=\sum_{l=0}^{N_L}\Sigma_{s,l}(\vec{r},E,E')\,P_l(\cos\theta)$

$\hat{\Omega},\cos\theta$

$$K\varphi=\int dE'\int d\hat{\Omega}'\sum_{l=0}^{N_L}\sum_{l'=0}^{N_L}\Sigma_{s,l'}(\vec{r},E,E')P_{l'}(\cos\theta)\sum_{m=-l}^{l}\varphi_l^m(\vec{r},E')Y_l^m(\hat{\Omega}')$$

$$=\int dE'\sum_{l=0}^{N_L}\sum_{l'=0}^{N_L}\sum_{m=-l}^{l}\Sigma_{s,l'}(\vec{r},E,E')\varphi_l^m(\vec{r},E')\int d\hat{\Omega}'P_{l'}(\cos\theta)Y_l^m(\hat{\Omega}')$$

$$\frac{4\pi}{2l'+1}Y_{l'}^m(\hat{\Omega})\delta_{ll'}$$

$$=\int dE'\sum_{l=0}^{N_L}\frac{4\pi}{2l+1}\Sigma_{sl}(\vec{r},E,E')\sum_{m=-l}^{l}\varphi_l^m(\vec{r},E')Y_l^m(\hat{\Omega})$$

Know Unknow. Know

Step 2: Discretize $E, \hat{\Omega}$

E: $\int dE \longrightarrow \Sigma_{s,l}(\vec{r}, E', E) \longrightarrow \underline{\Sigma_{s,g,g'}(\vec{r})}$

$HU(\vec{r}) \to \rho(\vec{r}) \to$ material $\to$ tabulated

$$\varphi(\vec{r}, \hat{\Omega}, E) \longrightarrow \varphi_g(\vec{r}, \hat{\Omega}) \qquad g_1 > g_2 > \cdots g_G$$

$\hat{\Omega}$: $\hat{\Omega}_{ij} = (\mu_i, \gamma_j)$ $\qquad \int d\hat{\Omega} f(\hat{\Omega}) = \sum_i \sum_j w_i w_j f(\hat{\Omega}_{i,j})$

$$\varphi(\vec{r}, \hat{\Omega}_{ij}) = \sum_{lm} \varphi_{lm} Y_l^m(\hat{\Omega}_{ij}) \Leftrightarrow \varphi_{lm} = \int d\hat{\Omega}\, Y_l^m(\hat{\Omega}) \varphi(\vec{r}, \hat{\Omega})$$

LBTE. $\hat{\Omega}_{ij} \cdot \vec{\nabla}\left[\varphi_{g'l}(\vec{r}) Y_l^m(\hat{\Omega}_{ij})\right] = S_g(\vec{r}, \hat{\Omega}_{ij}) + \Sigma_t \varphi(\vec{r}, \hat{\Omega}_{ij})$

$$\sum_{g'=1}^{N_g} \sum_{l=0}^{N_l} \frac{4\pi}{2l+1} \underset{\uparrow \text{Know}}{\Sigma_{sl}(\vec{r}, E, E')} \sum_{m=-l}^{l} \underset{\uparrow \text{unknow.}}{\varphi_{g'l}^{m}(\vec{r})} \underset{\uparrow \text{Know}}{Y_l^m(\hat{\Omega}_{ij})}$$

If one can solve: $\mathcal{L} \varphi_{g_1}(\vec{r}, \hat{\Omega}_{ij}) = \overset{\text{known.}}{S_{g_1}}(\vec{r}, \hat{\Omega}_{ij})$

Then: $\mathcal{L}_{22} \underset{\uparrow \text{unknown.}}{\varphi_{g_2}}(\vec{r}, \hat{\Omega}_{ij}) + \mathcal{L}_{12} \underset{\uparrow \text{Known}}{\varphi_{g_1}}(\vec{r}, \hat{\Omega}_{ij}) = \underset{\uparrow \text{known}}{S_{g_2}}$

$$\mathcal{L}_{gg'} = \hat{\Omega}_{ij} \vec{\nabla} \delta_{gg'} + K_{gg'}$$

Step 3: Dimensionality reduction : finite element

$$\tilde{\varphi}_S(\vec{r}) = \sum_{K=1}^{N_K} \varphi_K h_K(\vec{r}), \quad \varphi(\vec{r}) \in S^h; \quad h_K(\vec{r}) \in S^h$$

LBTE. $$\hat{\Omega}_{ij} \cdot \vec{\nabla}\left[\sum_{l=0}^{N_L} \sum_m \varphi_{gl}^m(\vec{r})\, Y_l^m(\hat{\Omega}_{ij})\right] = S_g(\vec{r}, \hat{\Omega}_{ij})$$

$$\sum_{g'=1}^{N_g} \sum_{l=0}^{N_L} \frac{4\pi}{2l+1} \underset{\text{know}}{\Sigma_{sl g'g}}(\vec{r}) \sum_{m=-l}^{l} \underset{\text{unknow.}}{\varphi_{g'l}^m}(\vec{r})\, \underset{\text{know}}{Y_l^m}(\hat{\Omega}_{ij})$$

Weak Form $$\int_{V_e} d\vec{r}\, \hat{\Omega}_{ij} \cdot \vec{\nabla}\left[\sum_{l=0}^{N_L} \sum_m \varphi_{gl}^m(\vec{r})\, Y_l^m(\hat{\Omega}_{ij})\right] g(\vec{r})$$

$$+ \int_{V_e} d\vec{r}\, S(\vec{r}, \hat{\Omega}_{ij})\, g(\vec{r}) + \int_{V_e} d\vec{r}\, K_{g'g}\left[\sum_l^{N_L} \sum_m \varphi_{gl}(\vec{r})\, Y_l^m(\hat{\Omega}_{ij})\right] g(\vec{r})$$

* stream term $$\int_{V_e} d\vec{r}\, \hat{\Omega}_{ij} \cdot \vec{\nabla}\left[\sum_{l=0}^{N_L} \sum_m \varphi_{gl}(\vec{r})\, Y_l^m(\hat{\Omega}_{ij})\right] g(\vec{r})$$

Boundary condition.

$$= \int_{V_e} d\vec{r}\, \hat{\Omega}_{ij}\, \varphi_g(\vec{r}, \hat{\Omega}_{ij}) \vec{\nabla} g(\vec{r}) + \int_{\partial V_e} g(\vec{r})\, \hat{\Omega}_{ij}\, \varphi_g(\vec{r}, \hat{\Omega}_{ij}) \cdot \hat{e}_n\, dS$$

$$= \int_{V_e} d\vec{r}\, \hat{\Omega}_{ij} \sum_{ml} Y_l^m(\hat{\Omega}_{ij}) \sum_{k=1}^{K_s} \varphi_{gK}^{lm} h_K(\vec{r}) \vec{\nabla} g(\vec{r})$$

$$+ \int_{\partial V} dS\, (\hat{\Omega}_{ij} \cdot \hat{e}_n) \sum_{ml} Y_l^m(\hat{\Omega}_{ij}) \sum_{k=1}^{K_s} \varphi_{gK}^{lm}\, g(\vec{r})\, h_K(\vec{r})$$

Pick $\hat{\Omega}_{ij}$, $\varphi(\vec{r}, \hat{\Omega}_{ij}, E)$ is reduced to $\left(\sum_{l=1}^{N_L} 2l+1\right) \cdot N_G, K_S$

✱ Scattering term.

$$\int_{V_e} d\vec{r} \sum_{g'=1}^{N_G} \sum_{l=0}^{N_l} \frac{4\pi}{2l+1} \Sigma_{slg'g}(\vec{r}) \sum_{m=-l}^{l} \varphi_{g'l}^{m}(\vec{r})\, Y_l^m(\hat{\Omega}_{ij})$$

$$= \sum_{g'=1}^{N_G} \sum_{l=0}^{N_l} \frac{4\pi}{2l+1} \Sigma_{slg'g}\cancel{(\vec{r})} \sum_{m=-l}^{l} Y_l^m(\hat{\Omega}_{ij}) \sum_{K=1}^{K_s} \varphi_{g'lK}^{m} \int_{V_e} d\vec{r}\, h_K(\vec{r})\, g(\vec{r})$$

✱ Absorb term + source term.

$$\int_{V_e} d\vec{r}\; S_g(\vec{r}, \hat{\Omega}_{ij}) + \Sigma_{tg}(\vec{r})\, \varphi_g(\vec{r}, \hat{\Omega}_{ij})$$

$$= \sum_{l=0}^{N_l} \sum_{m=-l}^{l} Y_l^m(\hat{\Omega}_{ij}) \sum_{K=1}^{K_s} S_{g'lK}^{m} \int_{V_e} d\vec{r}\, h_K(\vec{r})\, g(\vec{r})$$

$$+ \Sigma_{t,g} \sum_{l=0}^{N_l} \sum_{m=-l}^{l} Y_l^m(\hat{\Omega}_{ij}) \sum_{K=1}^{K_s} \varphi_{g'lK}^{m} \int_{V_e} d\vec{r}\, h_K(\vec{r})\, g(\vec{r})$$

✱ Boundary condition

$$\varphi_g(\vec{r} \in \partial V, \hat{\Omega}_{ij}) = \begin{cases} \varphi & \hat{\Omega} \cdot \hat{e}_n > 0 \quad \text{out} \\ \varphi_{inc} & \hat{\Omega} \cdot \hat{e}_n < 0 \quad \text{in.} \end{cases}$$

Solution to LBTE.

Internal source + External source

(primary beam, secondary beam, electron)

Boundary conditions ∂V: $\hat{\Omega}\cdot\hat{e}_n < 0$.

uncollided photon fluence φ^{unc}

analytical solution.

collided photon fluence

energy iteration.

finite element.

Boundary conditions.

φ^{inc} $\hat{\Omega}\cdot\hat{e}_n < 0$

φ $\hat{\Omega}\cdot\hat{e}_n > 0$

electron photon fluence.

energy ~~iteration~~

finite element.